MATH 245 F17, Exam 2 Solutions

1. Carefully define the following terms: predicate, $\forall x \in D, P(x)$, counterexample, Proof by Contradiction Theorem.

   A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain. The expression $\forall x \in D, P(x)$ is a proposition that is T if $P(x)$ is true for every $x \in D$, and F otherwise. A counterexample is an element of a domain that makes a predicate false. The Proof by Contradiction theorem states that for propositions p, q, if $(p \wedge \neg q) \equiv F$, then $p \to q$ is T.

2. Carefully define the following terms: Nonconstructive Existence theorem, Proof by Induction, Proof by Reindexed Induction, Proof by Strong Induction.

   The Nonconstructive Existence theorem states that if $(\forall x \in D, \neg P(x)) \equiv F$, then $\exists x \in D, P(x)$ is true. To prove $\forall x \in \mathbb{N}, P(x)$ by induction, we prove both that $P(1)$ is true, and that $\forall x \in \mathbb{N}$, $P(x) \to P(x+1)$ is true. To prove $\forall x \in \mathbb{N}, P(x)$ by reindexed induction, we prove both that $P(1)$ is true, and that $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \to P(x)$. To prove $\forall x \in \mathbb{N}, P(x)$ by strong induction, we prove both that $P(1)$ is true, and that $\forall x \in \mathbb{N}$, $P(1) \wedge P(2) \wedge \cdots \wedge P(x) \to P(x+1)$ is true.

3. Prove that for all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

   Proof by (ordinary) induction on n.
   Base case ($n=1$): The LHS has just one summand, namely 1. The RHS is $\frac{1(2)}{2} = 1$.
   Inductive case: Assume that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The next summand is $n+1$, which we add to both sides, to get $\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i = (n+1) + \frac{n(n+1)}{2} = (n+1)(1+\frac{n}{2}) = (n+1)\frac{2+n}{2} = \frac{(n+1)(n+2)}{2}$.

4. Prove or disprove: $\forall x \in \mathbb{Z}$, $|7x+20| > 1$.

   The statement is false. A counterexample is $x^\star = -3$, for which $|7x^\star + 20| = |-21+20| = |-1| = 1$, which is not strictly greater than 1. In fact, this happens to be the only counterexample.

5. Prove or disprove: $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ x^2 < y^2 < x^2+1$.

   The statement is true. Let $x \in \mathbb{R}$ be arbitrary. We must choose y, based on a side calculation. One possible choice is $y = \sqrt{x^2 + \frac{1}{2}}$. Now $y^2 = x^2 + \frac{1}{2}$, and since $x^2 < x^2 + \frac{1}{2} < x^2 + 1$, we get $x^2 < y^2 < x^2 + 1$.

6. Prove or disprove: $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R},\ x^2 < y^2 < x^2+1$.

   The statement is false. To disprove, we let $y \in \mathbb{R}$ be arbitrary. We must now choose x, based on a side calculation, to falsify $x^2 < y^2 < x^2 + 1$. One possible choice is $x = y$. This falsifies $x^2 < y^2$, and hence $x^2 < y^2 < x^2 + 1$ (which means $(x^2 < y^2) \wedge (y^2 < x^2 + 1)$).

7. Let F_n denote the Fibonacci numbers. Prove that $\forall n \in \mathbb{N}$, $F_{2n} = \displaystyle\sum_{i=0}^{n-1} F_{2i+1}$.

   This is proved with (ordinary) induction on n.

Base case ($n = 1$): The LHS is $F_2 = 1$, while the RHS is a single summand, namely $F_1 = 1$. Inductive case: Assume that $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$. The last summand is $F_{2(n-1)+1} = F_{2n-1}$. The next summand will be F_{2n+1}, so we add this term to both sides, to get $\sum_{i=0}^{n} F_{2i+1} = F_{2n+1} + \sum_{i=0}^{n-1} F_{2i+1} = F_{2n+1} + F_{2n} = F_{2n+2}$, where we used the Fibonacci recurrence to conclude that $F_{2n+1} + F_{2n} = F_{2n+2}$.

8. Let $x \in \mathbb{R}$. Prove that $2\lfloor x\rfloor \le \lfloor 2x\rfloor \le 2\lfloor x\rfloor + 1$.

   Solution 1: By a theorem (5.18) in the book, $\lfloor x\rfloor + \lfloor y\rfloor \le \lfloor x+y\rfloor \le \lfloor x\rfloor + \lfloor y\rfloor + 1$. Now set $y = x$ to get $\lfloor x\rfloor + \lfloor x\rfloor \le \lfloor x+x\rfloor \le \lfloor x\rfloor + \lfloor x\rfloor + 1$; the desired result follows.
   Solution 2: Since $x \ge \lfloor x\rfloor$, also $2x \ge 2\lfloor x\rfloor$. We apply a theorem (5.16) in the book to conclude that $\lfloor 2x\rfloor \ge \lfloor 2\lfloor x\rfloor\rfloor = 2\lfloor x\rfloor$, since $2\lfloor x\rfloor \in \mathbb{Z}$. Similarly, since $x \le \lfloor x\rfloor + 1$, also $x + x \le x + \lfloor x\rfloor + 1$, so we again apply theorem 5.16 to conclude that $\lfloor x+x\rfloor \le \lfloor x + \lfloor x\rfloor + 1\rfloor = \lfloor x\rfloor + \lfloor x\rfloor + 1$, by another theorem (5.17).

9. Let $n \in \mathbb{N}$. Prove that there is at most one $a \in \mathbb{N}$ satisfying $a^2 \le n < (a+1)^2$.

   Suppose that $a, b \in \mathbb{N}$ with $a^2 \le n < (a+1)^2$ and also $b^2 \le n < (b+1)^2$. We have $a^2 \le n < (b+1)^2$; taking square roots, we conclude that $a < b+1$. Similarly, we have $b^2 \le n < (a+1)^2$; taking square roots, we conclude that $b < a+1$ and hence $b - 1 < a$. Combining, we get $b - 1 < a < b + 1$. Applying a theorem from the book (1.12), since a, b are integers, we conclude that $a = b$.

10. Prove that $\sqrt{5}$ is irrational.

    We argue by way of contradiction. We suppose that $\sqrt{5}$ is rational. We can then express $\sqrt{5} = \frac{a}{b}$ where a, b are both integers, $b \ne 0$, and a, b have no factors in common. Squaring both sides, we get $5 = \frac{a^2}{b^2}$ and hence $5b^2 = a^2$. Thus $5|a^2$. Since 5 is prime, in fact $5|a$. Hence there is some integer c with $a = 5c$. We substitute to get $5b^2 = a^2 = (5c)^2 = 25c^2$. Dividing by 5 we get $b^2 = 5c^2$. Hence $5|b^2$, and since 5 is prime in fact $5|b$. But now a, b have 5 in common as a factor, which is a contradiction.